EXPERIMENT 20

Measurement of g Using the Kater Pendulum

Introduction

One of the most important physical constants is the acceleration due to gravity. It defines the unit of force for mechanics and consequently underlies all mechanical measurements. The value of the acceleration due to gravity can be measured directly to a reasonable accuracy (about 1%), although a well-conducted experiment on a free-fall apparatus or on a good air track will usually not be quite this accurate. To improve the accuracy, other methods must be used. In 1817 Kater, following a suggestion of Bessel, developed a reversible pendulum that made possible the accurate measurement of gravity. A reversible pendulum is a pendulum that can be swung from either of two pivot points. When the mass distribution of the pendulum is adjusted so that the periods are the same from either pivot, then this period is the same as a simple pendulum having a length equal to the distance between the pivots. While this may seem to be a simple property, it has great power in determining the value of "g" to high accuracy. Kater developed a form of pendulum that permitted the adjustment of the period with the pendulum swinging from the top or bottom pivot point. The pivots were knife edges, so that the distance between them could be measured with accuracy. Using pendulums of this type, the National Bureau of Standards determined the value of "g" to be 980.080 ±.003 cm/sec^2 in Washington in 1936. Similar precise measurements were made in England at Teddington and at Potsdam in Germany at about the same time. During a typical three-hour laboratory session, a student should be able to measure "g" to an accuracy of ±0.1%.

Theory

The mechanical design of the pendulum is not important to the derivation of the theory; it is only necessary that the pendulum can be swung from the two pivot points. The actual form of Kater's pendulum consists of a long bar with two knife edges placed near the ends. Attached to the bar are two bobs that can be adjusted on the bar to find the point of equal period. There is no single bob position for the equal period requirement to be met. Rather, the one bob can be positioned to make the periods equal for a range of positions of the other bob. The resulting period of the pendulum will be the same, and will depend on the distance between the knife edges and the value of "g".

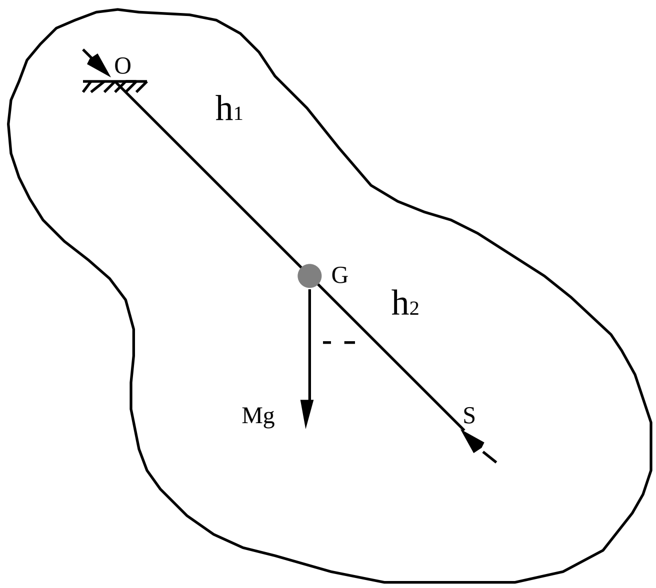

Figure 1

A generalized compound pendulum is shown in Figure 1. The pivot of the pendulum is at O and the center of mass at G. If this pendulum is moved from rest, the restoring couple is Mgh sinθ. Upon release of the pendulum, the motion will be described by

$$I_0 \,(d^2\theta/dt^2) = - Mgh_1 \sin\theta$$

where I_0 is the Moment of Inertia of the pendulum about the axis O.

If θ is small, the period is

$$T = 2\pi\sqrt{I_0 / Mgh_1}$$

The parallel axis theorem states that

$$I_0 = I_g + Mh_1^2$$

where I_g is the Moment of Inertia about an axis at the center of gravity, so that

$$I_0 = Mk^2 + Mh_1^2$$

where k is the radius of gyration of the pendulum about an axis at point G, the center of gravity.

Then $T = 2\pi\sqrt{(k^2 + h_1^2)/gh_1}$

$= 2\pi\sqrt{(h_1 + k^2/h_1)/g}$

Note that the radius of gyration is **defined** by

$$k^2 = Ig/M = \frac{1}{M}\int r^2\,dm$$

If the second pivot is located at S, so that the distance

$$GS = k^2/h_1 \quad ,$$

the period is then

$$T = 2\pi\sqrt{(OG + GS)/g} = 2\pi\sqrt{OS/g}$$

the same as a simple pendulum of length OS.

Inverting this pendulum, the period is given by

$$T = 2\pi\sqrt{I_S/Mgh_2}$$

where Is is the Moment of Inertia about the point S. But

$$I_s = I_g + Mh_2^2 = Mk^2 + M(k^2/h_1)^2$$

because

$$h_2 = k^2/h_1$$

by definition.

So that

$$T = 2\pi\sqrt{(Mk^2 + M(k^2/h_1)^2/(Mgk^2/h_1)}$$

$$2\pi\sqrt{(h_1 + (k^2/h_1))/g}$$

but

$$h_1 + k^2 / h_1 = OS$$

so the period is the same whichever pivot is used, and the period is the same as a simple pendulum of length OS, the distance between the knife edges.

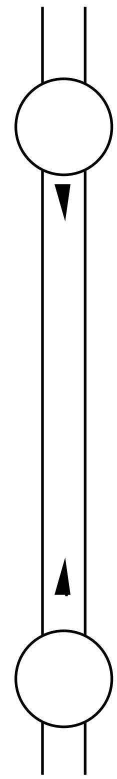

Figure 2

Experimental Procedure

The mounting bracket for the pendulum must be attached to a rigid wall; strong enough so there is a negligible motion of the mount when the pendulum swings. Even the slightest flexibility of the support will give an incorrect period for the pendulum. Attachment to the masonry wall is most satisfactory, although firm connection to a structural part of the building will serve almost as well.

The configuration of the pendulum is shown in Figure 2.

1. Note that the orientation of the knife-edges relative to the pendulum can be changed by means of small adjusting screws. Note also that the inclination of the wall-mounting "vee" can be changed by means of an adjusting screw. The pendulum should be mounted so that it oscillates about a horizontal axis. The alignment should be made in two parts. Firstly, the knife edges should be adjusted so that the position at the bottom of the pendulum is the same when it is turned through 180° about its long axis and replaced in the supporting "vee". The knife edges will then be perpendicular to the long axis of the bar. Secondly, the supporting "vee" should be adjusted so that the pendulum hangs vertically. This can be checked with a plumb bob.
2. Remove any rust or paint that may be in the "vee". The surface should be bare steel. Corrosion can be avoided by lightly oiling the surface. Rust or paint on the surface will make the knife edge pivots less efficient and absorb energy from the pendulum.
3. Carefully put the pendulum into the mount and verify that is swings freely and vertically. Make any further adjustments that may be required.

The two bobs on the pendulum can be placed on the bar in a number of ways. Investigating the options is interesting but time consuming and it is suggested that the bobs be placed on the bar shown in Figure 2. The value of "g" determined by the experiment is unaffected by the position of the bobs, so long as they are adjusted to meet the condition that the period from one knife edge is the same as the period from the other.

The first part of the experiment is to move one of the sliding bobs through a wide range to observe the change of period of the pendulum. After finding a convenient place for one bob, a series of measurements are made adjusting the other bob. From the series, the period of the pendulum is measured with accuracy with the second bob at three points near the equal period point.

I. Effect of the Position of the Bob

This part should be done quickly with help from the TA. If the approximate position of the equal period position is known then skip this part.

The pendulum is assembled so that one bob is outside of the knife edge. The other bob should be placed between the knife edges. The position of the bobs will be measured from the end of the bar to the edge of the bob. This is the easiest and most accurate measurement to make. The objective is to position the bob reproducibly.

4. Set one bob outside of the knife edge close to the end of the bar Tighten the screws to fasten the bob in place. This will be the "A" end of the pendulum.
5. Set the other bob between the knife edges near the "A" end. Position the bob so that it is 30 cm from the end of the bar. Tighten the thumb screw to hold it in place.
6. Gently set the knife edges in the "vee". Care is required because the knife edges are hardened and the "vee" is not, so the "vee" can be dented by the knife edges, or the knife edges chipped. Set the pendulum swinging so that its total motion is about 5 cm. Let the pendulum swing for half a minute to settle down.

There are many methods of measuring the period. Historically, the number of swings were counted between coincidences, comparing Kater's pendulum and the pendulum of a standard clock. While accurate, this method is very time consuming. Electronic timing methods are far faster and generally more accurate.

7. Measure the period of the pendulum in both the "A" up and "A" down positions. For this survey part of the experiment, the period is needed to ±.01s.
8. Move the second bob to 40 cm from the bar end and repeat the measurement of both periods.

Take measurements in 10 cm steps until the slider is at the far end of the bar. One of the measurements must be omitted to avoid the knife edges. Take the slider off and replace it outside of the knife edges.

A typical set of results are shown in Figure 3.

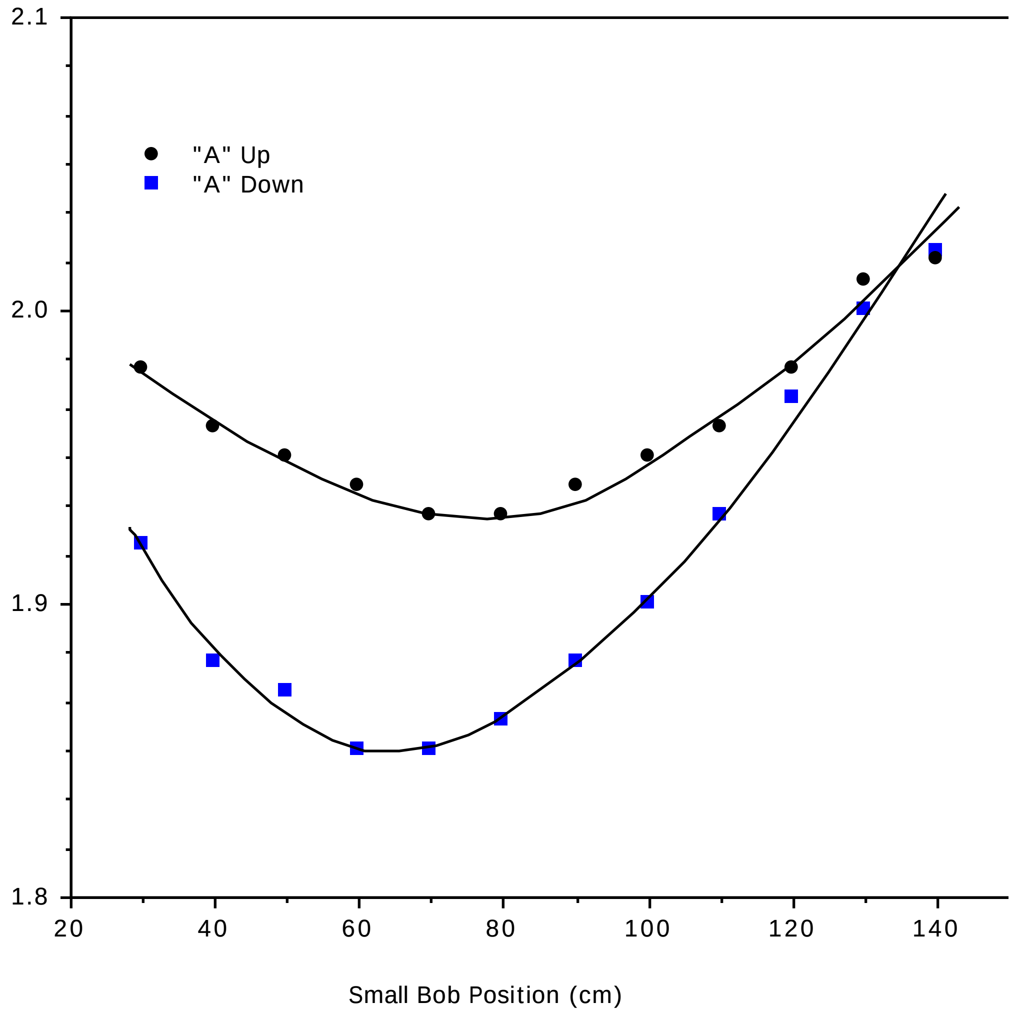

Figure 3

From these curves, it can be seen that the only place where the periods are equal is near 140 cm and the "A" end. The next part of the experiment will investigate this region.

<u>II. Measurements about Equal Period Position</u>

In this part, the slider will be moved in smaller steps about the assumed equal period point. Five or six positions of the slider will be measured and the period with "A" up and "A" down will be plotted. Near the equal period point, straight lines fit the data very well. The intersection of the "A" up line with the "A" down line is the equal period we are seeking. Accuracy can be improved by fitting "least square" lines to the data and solving for the intersection.

Since measuring the position of the slider from the "A" end is a nuisance, because it is so far away, we will make a change in coordinates and start measuring from any other convenient position. The position of the slider will be the same, it is just measured from a different place. Use calipers to measure the position.

9. Position the slider close to the equal period point. Determine both periods. Set the pendulum with "A" up and start it with a total amplitude of 5 cm.
10. Time a one-minute interval; single measurements with the timer are sufficient at this point. Turn the pendulum over and repeat the measurement with "A" down.
11. Move the slider two or three cm toward the knife edge and repeat the measurements. You will see that the two periods are almost equal throughout this region and that the "A" up period intersects with "A" down period in this region.

Plot these periods as a function of slider position. A typical plot is shown in Figure 4. The intersection of the lines gives T = 2.0068 s. This value was actually obtained by a "least squares" fit, rather than from the plot.

The second pair of lines on Figure 4 shows the corresponding results obtained when the first bob was in a different position. The lines are similar and their intersection is at about the same period as it must be, but the angle between the lines is greater. This improves the accuracy of the solution and, since accuracy is our objective, the experiment should be repeated for this position.

12.

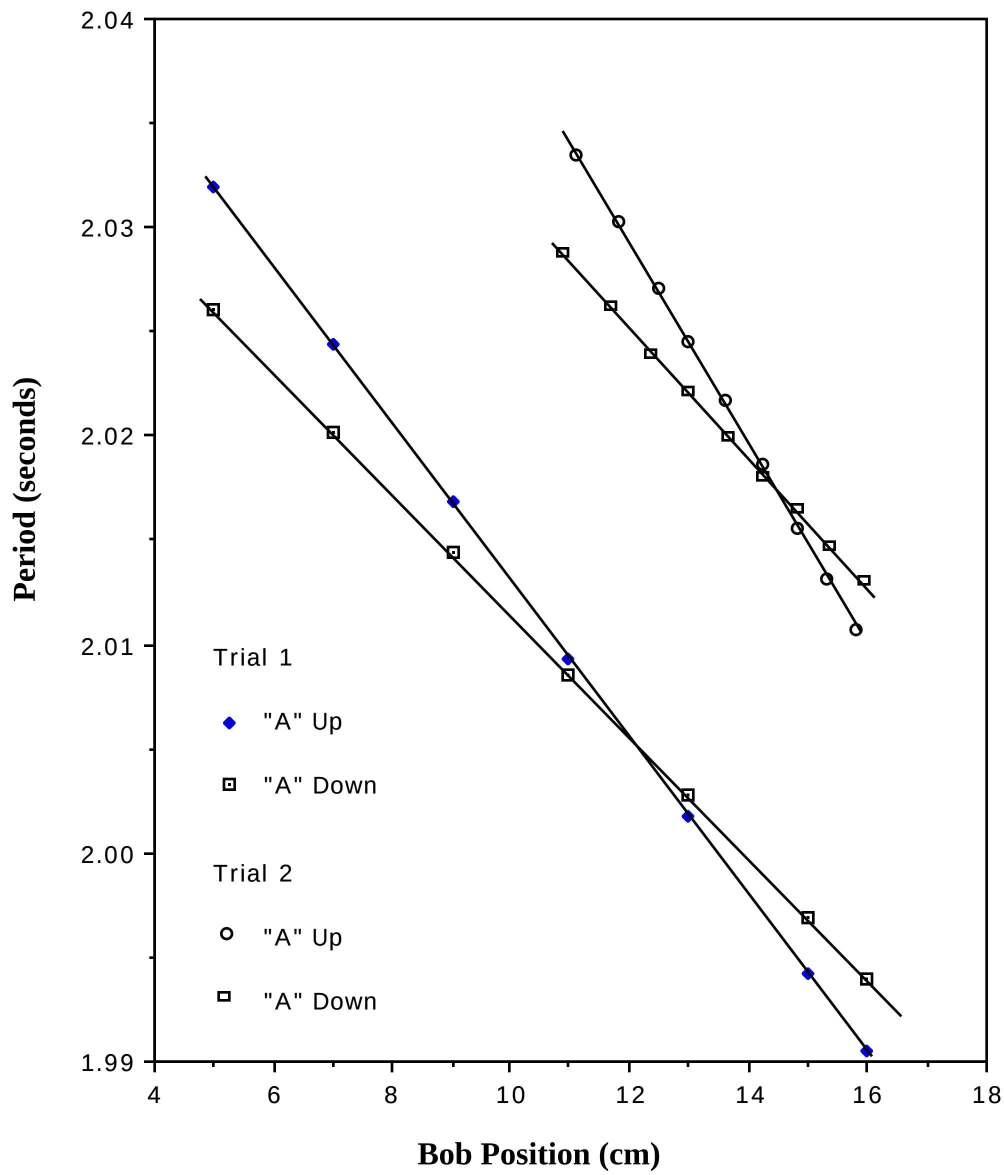

Figure 4

Plot the results and determine the equal period value. The period should be approximately equal to the previous value. For the data plotted in Figure 4, T = 2.0078 s. The difference represents the error remaining in the measurement. To improve the precision of the value, we must measure the position of the slider more accurately and improve our measurement of the period. The position of the slider can be accurately determined by measuring from the edge of the bob to the back side of the knife edge with calipers. This should give an accuracy of ±.01 cm depending upon the calipers used. Unfortunately, this makes another change in coordinates for the position measurement of the slider. However, keep in mind that the slider position needs to be known reproducibly, but not in absolute terms, so these coordinate shifts are confusing but not significant to the results.

13. Move the first bob a few cm and repeat steps 10 and 11.

To improve the accuracy of the time, more observations must be made, or a longer timing period used if swing counting is being used. Several observations will be made at each position. These data will establish the standard deviation of the time observations and assist in estimating the accuracy of the final value of "g" determined in the experiment.

There are two corrections that should be made to improve this result: measuring the actual separation between the knife edges, and checking the rate of the clock in the Counter Timer. Both of these parameters are constructed to close tolerances, but the requirements of this experiment exceed them. Let us discuss the knife edge separation first.

III Measurement of the Distance Between Knife Edges

There are a number of methods of performing this measurement, depending upon the equipment available. A measuring microscope or a cathetometer are the most convenient tools, even if their travel is limited and the total distance must be measured in steps. The screw on the table of the milling machine is accurate enough, if your machine shop has one. A steel tape or rule is not good enough, however, since the measurement must be accurate to .01 cm to be of any use.

For the pendulum used for the sample data, the average spacing between the two sides of the knife edges was 100.06 cm.

IV. Measuring the Period

The period of the pendulum may be measured using a digital counter. The timing apparatus consists of a pickup device, amplifier and counter. The pickup device is made up of a light source (infrared, diode) shining on a phototransistor (sensor). They are mounted across the end of a U-shaped holder (Fig. 5). The amplifier is connected to the counter "INPUT A" and a count is registered each time the pendulum crosses the light path.

You should always measure the period of oscillating objects relative to the most rapid portion of the motion. This minimizes the error in judging when a cycle is complete. One can usually assume that the oscillation is stable; the period does not change from cycle to cycle. In that case, if the basic error in timing is ∂t, the error in period due to timing n cycles is ∂t/n. For large numbers of swings it may be impractical to actually count the number of swings. If the period is measured over some small number of swings, the period can be used to determine the number of swings over some larger timing of cycles where the number of swings has not been counted. Of course, you must show that the number computed is within a small error of being a unique integer.

To measure the period of the pendulum it is necessary to mount the sensor and adjust the counter. Position the sensor by allowing the pendulum to hang without oscillating. Mount the holder so the bottom end of the pendulum is hanging between he poles of the horseshoe (Fig. 5). Adjust the holder so the light beam cuts one edge of the stationary pendulum. By eye, try to get close to the edge but do not labor the point. To measure the period of the pendulum, set the mode switch on the counter to "MULTI PERIOD A" and set the number of periods to 10. Set the sensitivity on the counter to 1 volt and swing the pendulum with an amplitude of about 4 cm. Increase the sensitivity vernier until the counter is registering properly. The counter should now record the time for 5 swings. It is important to note that the counter rolls over after 9999.99 ms.

V. Measuring the Clock Rate of the Counter Timer

The Timer is crystal controlled, so its clock rate is very stable and reproducible. It is not necessarily accurate enough for this measurement. It is quite simple to measure the rate and make the correction as needed.

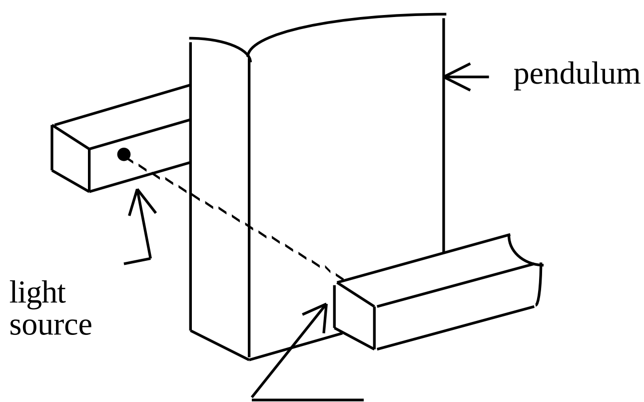

Figure 5

17. Using a well-calibrated stop watch, start the watch and release the reset button together. There may be a slight error doing this, of the order of .3 sec, but this becomes insignificant if the test period is long enough.
18. As a preliminary test, let the counter Timer run for 4000 seconds. When 4000 appears, stop the watch.

19. From the elapsed time as measured on the watch, calculate the Timer's clock rate.
20. With this preliminary value, extend the check to a much longer time interval; e.g., 6000 sec, and calculate the Timer's average clock rate.

For the Timer used for the sample data, I second on the display was actually .99971 seconds. This reduces the observed period to

$$T = 2.00703 \text{ seconds.}$$

Making the correction for K.E. separation and the correction for clock rate, gives a final value of

$$g = 980.6 \text{ cm/sec}^2$$

This compares with the standard value of "g" for Salem at Latitude 41.5180° of 980.316 cm/sec/sec. Local factors may cause the small difference between these values or it may represent the accuracy limit of the present experiment. You should compare your value with the best known value of local "g" to see how well your measurement compares.

Discussion

Once the position of the bobs for equal periods is determined, the pendulum can be moved from place to place to survey the variation of the acceleration due to gravity with location. The greatest variation of "g" is due to the latitude of the location. This latitude variation is due to two causes. The first is the difference in the radius of the earth with latitude, the earth being fatter at the equator. Since the gravitational force is inversely proportional to the square of the distance between the centers of mass, gravity is smaller at the equator. The second cause is the centrifugal force of the earth's rotation, which is a maximum at the equator and zero at the poles. Both effects are in the same direction, so "g" increases with latitude, being smallest at the equator. The Handbook of the American Institute of Physics gives the value of the acceleration as

$$g = 978.0490\,(1 + .0052884 \sin^2\phi - .0000059 \sin^2 2\phi)$$

where Ø is the latitude of the measurement point. The latitude of the location can be found accurately enough from a topographic map of the area. Local effects may cause small differences from the values predicted by this equation but to the level of accuracy to be expected from a normal laboratory experiment, this equation will predict the local value of "g" correctly.

Residual Errors in the Measurement

The first thing to check is the accuracy of the stop watch used to measure the period of the pendulum. Watches are remarkably accurate devices but the error in their rate may be significant in an experiment as accurate as this one. Accurate time can be obtained from many radio stations on the hour. More accurate and useful time signals can be obtained from WWV, the radio station of the National Bureau of Standards, which broadcasts standard time signals on short wave at 5, 10, 15 MHz throughout the day and night. A check of the watch used in the sample experiment showed that it was running slow at a rate of 5 seconds/day.

The separation between the knife edges would have to be measured with greater accuracy to improve the determination. The Bureau of Standards used an optical interferometer and measured the distance to a fraction of a wavelength.

More subtle factors must be considered, if accuracy beyond four significant figures is desired. To improve the timing accuracy, the beat of the pendulum can be measured over a much longer timing interval. This requires that damping of the pendulum, due to air drag as well as pivot friction, be reduced. The air drag can be eliminated by swinging the pendulum in a vacuum. Pivot friction can be reduced by using sapphire knife edges. The period measured for the Bureau of Standards pendulum in Washington was 2.004 454 seconds. In one experiment, the timed interval started at 08:35 and finished at 13:50.

For an interesting account of the measurements made by the National Bureau of Standards, reading the following paper is strongly recommended:

P.R. Heyl and G.S. Cook, **Value of Gravity in Washington,** J. Res. of Bureau of Standards, **17,** 805-839 (1936).